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Probability distribution for a multifractal field

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The complete probability distribution function for a multifractal field with quadratic $f(\alpha)$ is derived. We do this for a model which is obtained as the continuous limit of an infinite product of random functions each one having log-normal statistics. The resulting probability distribution is also log normal but with long-range logarithmic correlations. This can be written in a local $n \to 0$ form.

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I. INTRODUCTION

Many phenomena in condensed-matter physics appear to exhibit multifractal statistics. Systems where these statistics have been discovered include turbulence [1–3], diffusion near fractals [4], electrons in disordered media [5], polymers in disordered media [6], random ferromagnets [7], chaotic dissipative systems [8], and random resistor networks [9]. There are two basic properties [8] of a multifractal field $O(\mathbf{r})$. First it must be self-similar under coarse graining. Second, the qth moment of $O(\mathbf{r})$ must satisfy the following scaling law [10] when averaged over different realizations of the field O:

$$\langle O(\mathbf{r})^q \rangle \sim \left[\frac{a}{R} \right]^{x_q} . \tag{1}$$

Here R is a typical linear dimension of the system, x_q is the fractal dimension, and a is a small distance cutoff. $\mathbf{r}$ is kept constant during the averaging, and can be taken to be any position. x_q are the "scaling dimensions" of O and the values at different q are not related to each other in a trivial way. There is a linear relation [10] between the x_q's and the more familiar τ_q's used by Halsey *et al.* [8]. In the case of turbulence, the field $O(\mathbf{r})$ is thought to be the local energy dissipation. Multifractal statistics imply that different moments of the energy scale as if having different fractal dimensions.

Because they occur in many systems, there has been much effort devoted to giving theoretical explanations for multifractal phenomena. Most theoretical work has been for random disordered systems where the replica is frequently performed. There is still little known about how to relate multifractal scaling to standard critical phenomena, where Lagrangian field theory (FT) has been highly successful. In a recent paper [10], the connection between FT and multifractal statistics was explored. The scaling dimension for multifractal moments must be a convex function of its order, which is the opposite of what is found in FT's. It was suggested that field gradients may be a more appropriate candidate to connect multifractal scaling with FT's.

In this paper, we derive the expression for the complete probability distribution of a multifractal field having a quadratic $f(\alpha)$. This is a case of considerable practical interest because the quadratic approximation is quite good for a variety of systems and also because small $f(\alpha)$, the region of strongest departure from a quadratic form, is difficult to ascertain experimentally. The resulting distribution for our system is quite simple. It is log normal in the measure and has long-range logarithmic correlations. This result is useful in the context of a system subject to multifractal noise, or in modeling multifractal phenomena. A knowledge of the probability distribution is often quite important in the understanding of a random process.

The outline of this paper is as follows. We first introduce the "continuous scale model," which is the infinite product of a set of random functions, each one with a log-normal distribution. We then derive the probability distribution for this model and check that it obeys multifractal scaling.

II. CONTINUOUS SCALE MODEL

The simplest way to understand multifractal statistics is in terms of a "blob" picture. One continues to subdivide a system into smaller scales. At each scale one assigns new weights to these finer subdivisions. An example of this is the two-scale Cantor set [8], or the beta model of turbulence [3]. In the latter case, multifractal scaling is thought to apply to the local energy dissipation. The velocity field in turbulent flow can also be understood in terms of such a blob picture, but the statistics of the velocity field appear quite different than the energy dissipation and less is understood about them [11,12]. Here an idea related to the "blob" picture is employed which is more suitable for obtaining the complete probability distribution.

Consider a Gaussian random function $u(\mathbf{r})$. This is completely characterized by its mean and two-point correlation function

$$\langle u(\mathbf{r}) \rangle = \bar{u} , \tag{2}$$

$$\langle u(\mathbf{r})u(\mathbf{r}')\rangle = M(|\mathbf{r}-\mathbf{r}'|) \ . \tag{3}$$

We wish to consider $\ln\phi(\mathbf{r})\equiv u(\mathbf{r})$, so that ϕ is log-normally distributed. Now consider N realizations of $\phi(\mathbf{r})$ chosen at random from this distribution, label them $\phi_i(\mathbf{r})$ where $i=1,\ldots,n$. The blob constructions mentioned above correspond here to multiplying these functions together while rescaling each function at every scale

$$O(\mathbf{r}) = \prod_{i=1}^{N} \phi_i(e^{C_i}\mathbf{r}) \ . \tag{4}$$

In order to obtain a simple expression, we take the limit as $N\to\infty$ and $C\to 0$ in such a way as to give a well-defined limit. Thus we write

$$\ln O(\mathbf{r}) = \int_0^{\Gamma} u_\gamma(e^{c\gamma}\mathbf{r})d\gamma \ , \tag{5}$$

where γ characterizes the amount of rescaling. At the upper limit of integration, the function has been shrunk by a scale factor $\exp(c\Gamma)$. Writing out the probability distribution in full for all values of u and γ, corresponding to Eq. (3), one has

$$P(u_\gamma)\delta u_\gamma \propto \exp\left[-\tfrac{1}{2}\int_0^{\Gamma}\int\int[u_\gamma(e^{c\gamma}\mathbf{r})-\bar{u}]M^{-1}(\mathbf{r},\mathbf{r}')[u_\gamma(e^{c\gamma}\mathbf{r}')-\bar{u}]d^d\mathbf{r}\,d^d\mathbf{r}'d\gamma\right]\delta u_\gamma \ , \tag{6}$$

where d is the spatial dimension and the function M^{-1} is the continuous analog of the inverse matrix of $M(\mathbf{r},\mathbf{r}')\equiv M(\mathbf{r}-\mathbf{r}')$.

III. PROBABILITY DISTRIBUTION

With the model as defined above we now wish to derive the probability distribution of $\ln O$, $P\{\ln O(\mathbf{r})\}$. From Eq. (5) $\ln O$ is a linear combination of the u_γ's. Because the u_γ's are Gaussian random variables, then $\ln O$ must also be random. It is therefore characterized completely by its mean and two-point correlation function which we now compute

$$\overline{\ln O} = \langle \ln O\rangle = \left\langle \int_0^{\Gamma} u_\gamma(e^{c\gamma}\mathbf{r})d\gamma\right\rangle = \bar{u}\Gamma \ . \tag{7}$$

From Eq. (6) the cumulant two-point correlation function is

$$\begin{aligned} N(\mathbf{r}-\mathbf{r}') &\equiv \langle \ln O(\mathbf{r})\ln O(\mathbf{r}')\rangle_C \\ &= \int_0^{\Gamma}\langle u_\gamma(e^{c\gamma}\mathbf{r})u_\gamma(e^{c\gamma}\mathbf{r}')\rangle d\gamma \\ &= \int_0^{\Gamma} M(e^{c\gamma}|\mathbf{r}-\mathbf{r}'|)d\gamma \ , \end{aligned} \tag{8}$$

where the subscript C represents a cumulant average. Letting $r\equiv|\mathbf{r}-\mathbf{r}'|$ and $s\equiv r\exp(c\gamma)$ this becomes

$$\int_r^{r\exp(c\Gamma)} M(s)\frac{ds}{cs} \ . \tag{9}$$

If the correlation function $M(s)$ has a finite correlation length ξ, then the upper limit of this integral can be taken to infinity when $r \gg \exp(-c\Gamma)\xi$. Then the region around the lower limit will give rise to a logarithmic dependence on r, of the two-point function $N(\mathbf{r}-\mathbf{r}')\sim[M(0)/c]\ln(\xi/r)$, where here we have also assumed that $r\ll\xi$. As an example, choose

$$M(s) = \begin{cases} M_0(1-s) & \text{for } s<1 \ . \\ 0 & \text{otherwise} \ . \end{cases} \tag{10}$$

Then Eq. (9) gives

$$N(r) = \begin{cases} \dfrac{M_0}{c}\left[\ln\left[\dfrac{1}{r}\right]-1+r\right] & \text{for } 1>r>e^{-c\Gamma} \ , \\ \dfrac{M_0}{c}[c\Gamma-(e^{c\Gamma}-1)r] & \text{for } r<e^{-c\Gamma} \ . \end{cases} \tag{11}$$

As suggested above, the dominant term in the range $1>r>e^{-c\Gamma}$ is logarithmic. Thus we can conclude that the probability distribution for $\ln O$ is

$$P\{\ln O(\mathbf{r})\}\propto\exp\left[-\tfrac{1}{2}\int\int[\ln O(\mathbf{r})-\overline{\ln O}]N^{-1}(\mathbf{r},\mathbf{r}')[\ln O(\mathbf{r}')-\overline{\ln O}]d^d\mathbf{r}\,d^d\mathbf{r}'\right] \ . \tag{12}$$

The notation is the same as in the preceding section in that the function N^{-1} is the continuous analog of the inverse matrix of $N(\mathbf{r},\mathbf{r}')\equiv N(\mathbf{r}-\mathbf{r}')$.

This is the complete probability distribution corresponding to the continuous scale model. Because c can be eliminated through a rescaling of γ it will be set equal to 1 from now on.

The range of scales where fractal scaling should hold goes from the lower cutoff $a\sim\exp(-\Gamma)\xi$ to the largest length scale $R\sim\xi$. The corresponding $f(\alpha)$ can also be obtained. $f(\alpha)$ can be defined through the probability distribution at one point

$$P(\ln O)d\ln O \propto \exp\left[\ln\left[\frac{R}{a}\right]f\left[\ln O\Big/\ln\left[\frac{R}{a}\right]\right]\right]d\ln O \ . \tag{13}$$

From Eq. (12) the probability distribution at a point can be obtained. It must also be log normal and the variance is $N(0)=\Gamma M_0$. The corresponding $f(\alpha)$ is obtained by noting from above that $\Gamma=\ln(R/a)$, therefore

$$f(\alpha) = -\frac{(\alpha-\bar{u})^2}{2M_0} \ . \tag{14}$$

One can also check that the two-point correlation functions for this model obey the form proposed by Cates and Deutsch [13]. First note that by a linear shift in $\ln O$, the factor $\bar{u}$ can be eliminated, so we will consider the case $\bar{u}=0$ which simplifies the algebra in what follows. First we calculate the x_q's by computing $\langle O^q \rangle$. This gives $x_q = -(M_0/2)q^2$. Now we consider the correlation functions

$$\langle O(\mathbf{r})^p O(\mathbf{r}')^q \rangle = \langle e^{p \ln O(\mathbf{r}) + q \ln O(\mathbf{r}')} \rangle \; . \tag{15}$$

Because the probability distribution is quadratic in $\ln O(\mathbf{r})$ this integral can be done in closed form. Using the logarithmic correlation function obtained above, this is proportional to r^{-pqM_0}. This exponent is also equal to $x_{p+q} - x_p - x_q$, in agreement with earlier work.

IV. DISCUSSION

The relationship between the result found here and FT remains open. Duplantier and Ludwig [10] analyzed the distinction between systems giving rise to multifractal statistics and "standard" field theory. The exponents x_q have field-theoretic analogs in the form of scaling operators. Although there are formal similarities between statistics in critical phenomena and multifractal statistics, an essential stability requirement for exponents is reversed for a standard ϕ^4 theory and multifractals. There are several cases [4,7,6,14], however, of systems involving random critical behavior that show multifractal statistics and are qualitatively different for more standard field theories. In the cases that have been analyzed theoretically [4,7,14], the x_q are quadratic to first order in an ϵ expansion which implies a quadratic $f(\alpha)$ which is identical to the case considered in this paper. For example, Ref. [7] considered ferromagnetic spins with quenched disorder finding that the average magnetization varied from site to site in a way characterized by multifractal statistics.

The systems just mentioned can be understood by considering an effective Lagrangian that can be written as an expansion of a field and its gradients about zero. It is of interest to compare this to what we have found here. Equation (12) can be written in a local form in an even number of dimensions. For example, in two dimensions one has the Lagrangian

$$L = (\nabla \ln O)^2 \; . \tag{16}$$

The $N^{-1}(\mathbf{r},\mathbf{r}') = \nabla^2$ when one takes $N(\mathbf{r})$ to be logarithmic. In general, for even dimensions one obtains

$$L = \ln O (\nabla^2)^{d/2} \ln O \; . \tag{17}$$

The Lagrangian is highly nonlinear in the field O and in contrast to the cases above *cannot* be written as an expansion of the field O about $O=0$. This is to be expected as a "standard" field theory should not lead to multifractal statistics [10]. One approach to relate this to a more standard field theory may be via the often used identity $\ln O = \lim_{n\to 0}(O^n - 1)/n$. This means that one should study the Lagrangian

$$L = O^n (\nabla^2)^{d/2} O^n \tag{18}$$

as a function of n and look at the limit as $n \to 0$. Under renormalization additional terms are generated. At present it is not clear how to add terms to this Lagrangian so as to obtain a sensible limit as $n \to 0$.

However, we have shown that it is possible to write down a highly nonlinear field theory that gives multifractal statistics and therefore has the opposite convexity as found in most standard field theories.

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